



An Analysis of a Bounded Resource Search puzzle

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History

First posed in Foundation Of Computer Science - FOCS 1980

Algorithms & Theory of Computation book has an algorithm(non-optimal)

Several variations were studied.

My paper, published in Arxiv.org (May 2004) – First paper to study the general variant solution using Dynamic Programming.

<https://arxiv.org/abs/cs/0405110>

- Mathematics Magazine – ‘The Egg-Drop Numbers’ – Vol. 77. No . 5 (Dec., 2004), pp. 368-372. possible independent study with rigorous formal proof.

Several sites with solution – Applicability? Not realized, as much!

A deceptively simple Puzzle

- Building with n floors. Given k glass balls, find the lowest floor from which if you throw, a glass ball breaks. Glass balls being uniform in physical properties. Let m be minimum number of tries. Let L be the index of such a floor.
- Variant is Eggs instead of Glass
- Famous CS interview question for $k = 2$ at Borland in the 90s to join Delphi/Turbo Pascal Group!
- Mathematical Magazine Dec 2004 paper contains rigorous proof. Title : “An Egg-drop problem” !

Yeah! The real puzzle is to Optimize Definitions first,
before running Theorem mills...

Main Result

- For $k = 2$, $m = \left\lceil \frac{\sqrt{8n+1}-1}{2} \right\rceil$ is well known.
- Main Result :

$$\text{For any } k, \quad 1 + \sum_{i=1}^k \binom{m-1}{i} < n \leq \sum_{i=1}^k \binom{m}{i}$$

Proof for $k = 2$

- Proof using simple counting!
- Let $\{p_1, p_2, p_3 \dots p_m\}$ be the final set of floor indices where an attempt is made. Assuming, L exists and $p_0 = 0$.
- Assume the ball did not break until p_{i-1} . Next attempt at p_i the ball breaks. There are $m - i$ attempts left, $p_i - p_{i-1} - 1$ floors left, 1 ball left. *Therefore, $m - i = p_i - p_{i-1} - 1$.*
- Note that When p_m is the last floor, Total no. of floors - which is simply all the probed floors & floors between them i.e. $n = m + \sum_{i=1}^m p_i - p_{i-1} - 1 = m(m + 1)/2$ applying previous eqn.

From which we get $m = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor$.

Proof for any *positive* k

- Proof using dynamic programming!
- Suppose $\{p_1, p_2, p_3 \dots p_m\}$ be the final set of floor indices where an attempt is made. Assuming, L exists and $p_0 = 0$.
- Suppose there exists a function $P: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ s.t.

$$P(m, k) \geq n > P(m-1, k) \text{ and}$$

$$P(0, i) = 0 = P(i, 0)$$

The maximum value of this function, using dynamic programming principles of optimal sub-problem, is .

$$P(m, k) = m + \sum_{i=1}^{m-1} P(i, k-1)$$

(Interesting variation is $P(m, k) = m + \prod_{i=1}^{m-1} P(i, k-1)$)

Proof for any *positive k* contd...

- So, $P(m-1, k) = m-1 + \sum_{i=1}^{m-2} P(i, k-1)$

Subtracting, we have the familiar,

$$P(m, k) = 1 + P(m-1, k) + P(m-1, k-1)$$

a non-irreducible Solution is,

$$P(m, k) = \sum_{j=1}^k \binom{m}{j}$$

No Closed form!

So, for $k=2$

$$\sum_{j=1}^2 \binom{m}{j} = \binom{m}{1} + \binom{m}{2}$$

Just set this equal to n . Solve for m

$$\frac{m!}{(m-1)!} + \frac{m!}{(m-2)!} = m + \frac{m(m-1)}{2} \Rightarrow \frac{m(m+1)}{2} = n, \text{ same as before!}$$

If $k=m$, We have now,

$$\sum_{j=0}^m \binom{m}{j} = 2^m = \sum_{j=1}^m \binom{m}{j} + \binom{m}{0} = n + 1$$

which means $m = \log_2(n + 1)$. So $n \rightarrow \infty, m \rightarrow \log_2(n + 1)$

The Ubiquitous Binary Search for $k = m$ and above!

So, which floors to try?

In the worst case a glass ball will break at a floor 'before' the first floor we try p_1 ... We use that fact to find p_1 .

$$p_1 - p_0 - 1 = P(m - 1, k - 1)$$

$$\Rightarrow p_1 = 1 + P(m - 1, k - 1)$$

$$\Rightarrow p_1 = 1 + \sum_{j=1}^{k-1} \binom{m-1}{j}$$

Then on, p_i is dependent on whether the ball broke at the previous floor tried or not.

$$p_i = \begin{cases} p_{i-1} + 1 + P(m - 1, k - 1), \\ p_{i-1} + 1 + P(m - 1, k) \end{cases}$$

Examples for k= 3, 4, 5

 WolframAlpha

real roots of (M choose 1) + (M choose 2) + (M choose 3) = 100

 NATURAL LANGUAGE  MATH INPUT  EXTENDED KEYBOARD  EXAMPLES  UPLOAD  RANDOM

Input interpretation

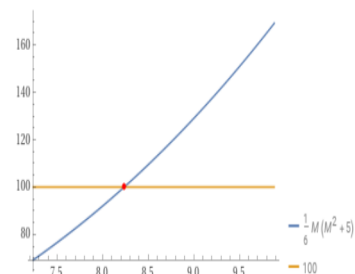
solve $\binom{M}{1} + \binom{M}{2} + \binom{M}{3} = 100$ over the reals

$\binom{n}{m}$ is the binomial coefficient

Result Approximate form ☒ Step-by-step solution

$$M = \frac{\sqrt[3]{5(540 + \sqrt{291615})}}{3^{2/3}} - \frac{5^{2/3}}{\sqrt[3]{3(540 + \sqrt{291615})}}$$

Plot



 WolframAlpha

real roots of (M choose 1) + (M choose 2) + (M choose 3) + (M choose 4) = 100

 NATURAL LANGUAGE  MATH INPUT  EXTENDED KEYBOARD  EXAMPLES  UPLOAD  RANDOM

Input interpretation

solve $\binom{M}{1} + \binom{M}{2} + \binom{M}{3} + \binom{M}{4} = 100$ over the reals

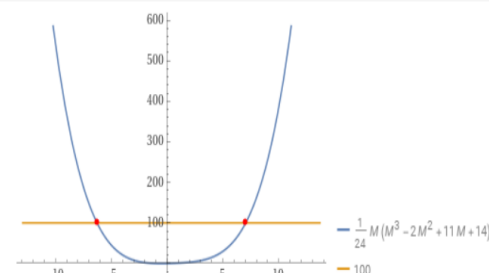
$\binom{n}{m}$ is the binomial coefficient

Results More digits Exact forms ☒ Step-by-step solution

$M \approx -6.2825$

$M \approx 7.0382$

Plot



Number line



FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA

 WolframAlpha

real roots of (M choose 1) + (M choose 2) + (M choose 3) + (M choose 4) + (M choose 5) = 100

 NATURAL LANGUAGE  MATH INPUT  EXTENDED KEYBOARD  EXAMPLES  UPLOAD  RANDOM

Input interpretation

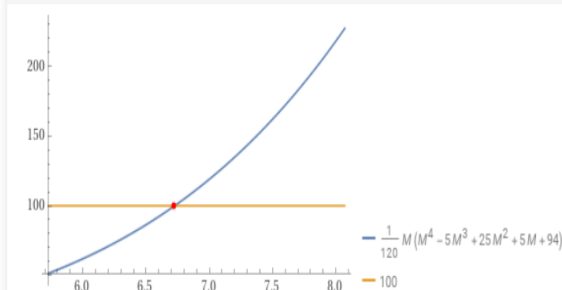
solve $\binom{M}{1} + \binom{M}{2} + \binom{M}{3} + \binom{M}{4} + \binom{M}{5} = 100$ over the reals

$\binom{n}{m}$ is the binomial coefficient

Result Enlarge Data Customize Plain Text

$M \approx 6.72629$

Plot



Number line

Psst: Ahem!, I...However have a bunch of eggs & NaNs



Quick look at the Polynomial-in-hand

$$\sum_{j=1}^k \binom{m}{j} = n$$

For Larger n , Note integral coefficients...

We see Diophantine equations, binomial symmetry..

Are there real roots? Is it Solvable?

Not sure for your n or m or k ?

Galois Theory applies & attempt to see if there is a solution, possibly ! Approximate algorithms may work as well..

Variations and Extensions

You may know m , k but not n etc. Clearly two entities must be known.
 n varies with time or depending on result on a floor you change n (Network loading)

Continuous scenario. What if height of each floor ' dx ' is infinitesimal?
Then $n \rightarrow \infty$. Binary search is the way...

Balls, meaning of 'glass breaking' is 'loosely defined' for enough applications.

e.g. ignore some floors if glass breaks f_1 , f_2 etc. there are some spare balls those floors!. Break condition is if $>$ something or less than etc.
Leading to applications...

Applications, A lot!

Here is Four more besides Network Loading...

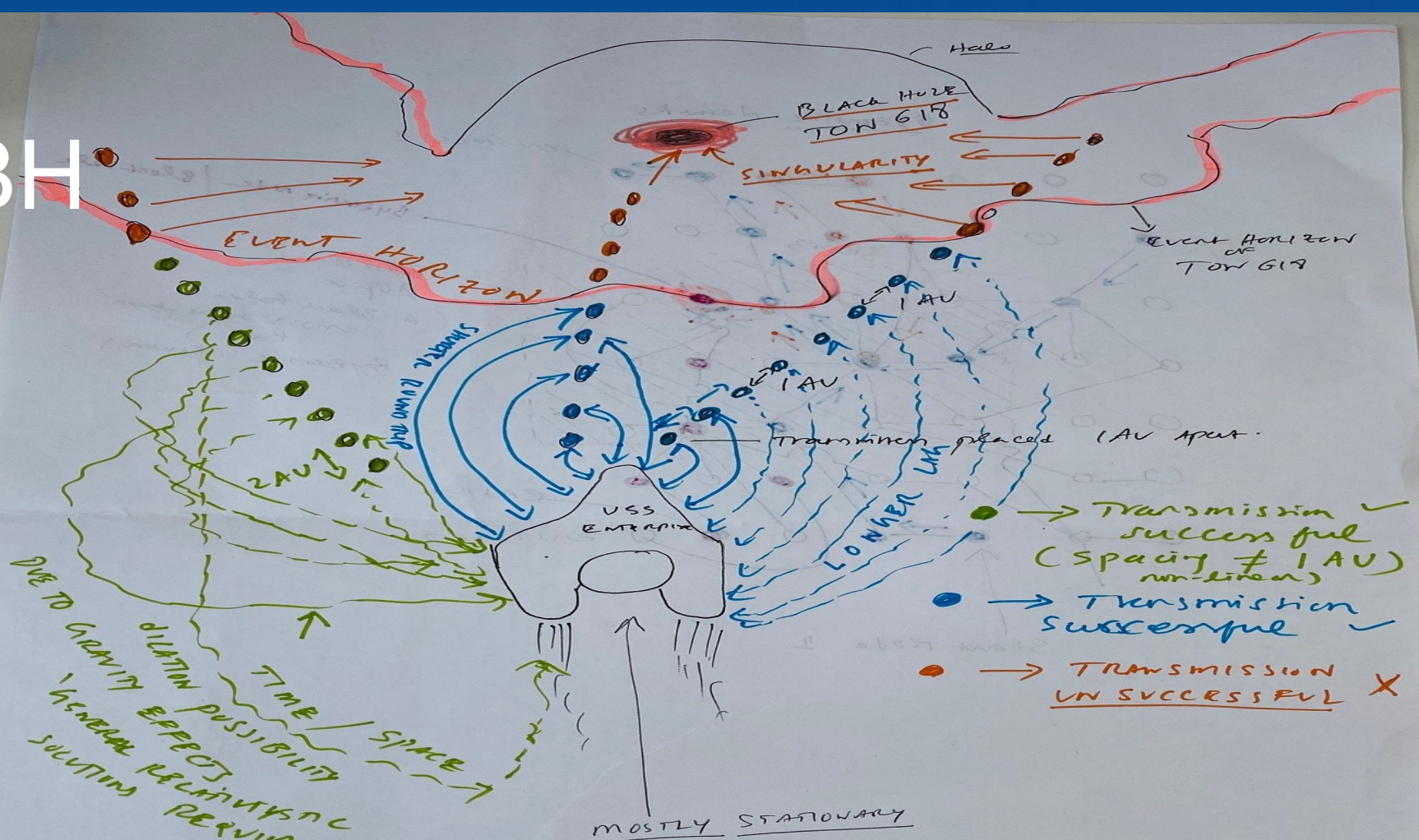
Real Blackhole Event Horizon advance measure...

Data stream with long sequence of 0s starting 'somewhere' and 0s after.

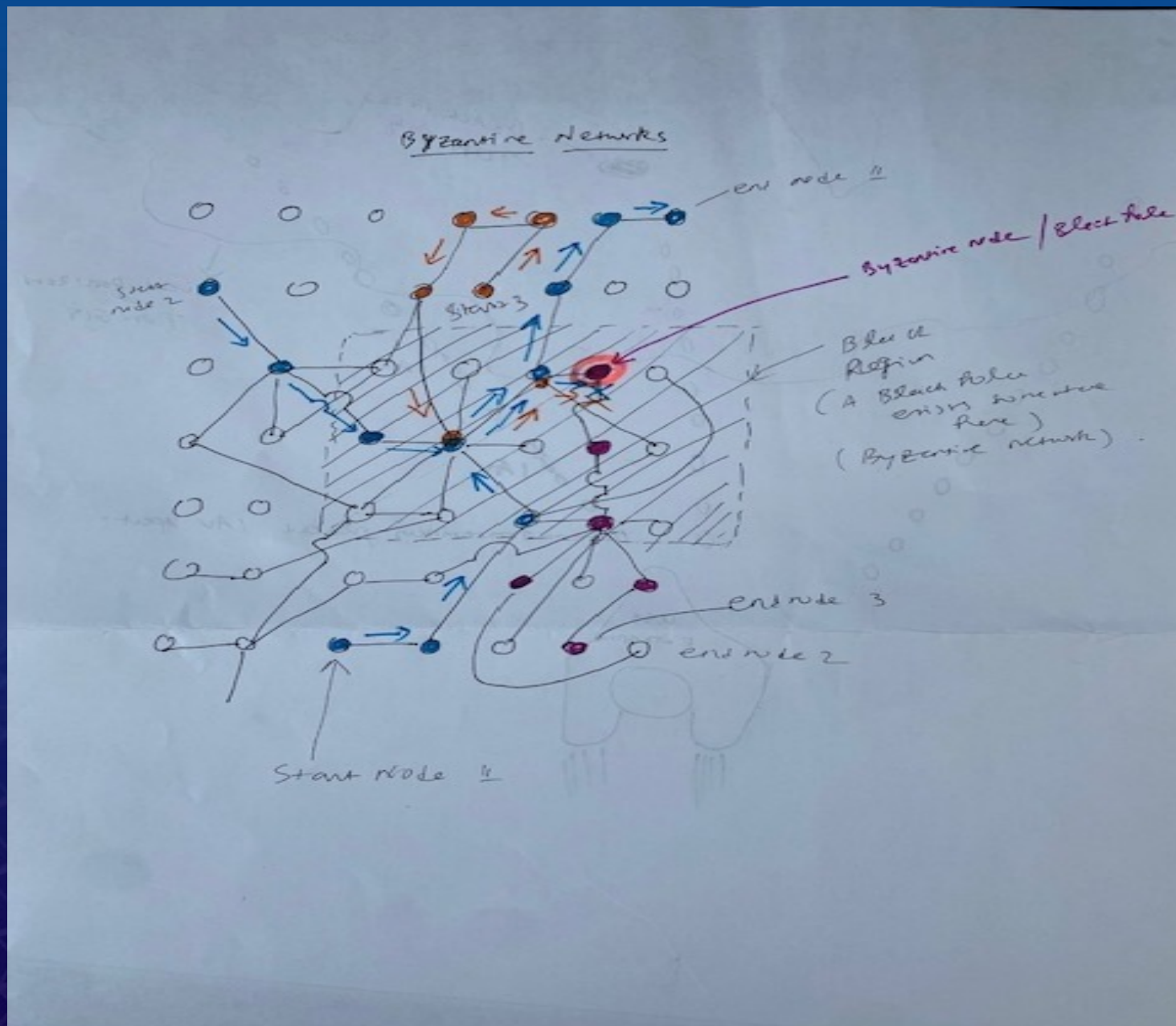
Malicious 'Node', eating Web Crawlers – called Blackholes, in Byzantine Networks

Minima search in Convex models

BH



- → FOR THE GRAVITATIONAL PERTURBATIONS THE TRANSMITTERS ARE PLACED POSSIBLY ON A CURVE OR ON A 'HERMITE PATCH' LIKE STRUCTURE. THIS IS TO ENSURE UNIFORM & ORDERLY & LINEAR SIGNAL RECEPTION.





Thank You!